

Fano type quantum inequalities in terms of q -entropies

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Generalizations of the quantum Fano inequality are considered. The notion of q -entropy exchange is introduced. This quantity is concave in each of its two arguments. For $q \geq 0$, the inequality of Fano type with q -entropic functionals is established. The notion of coherent information and the perfect reversibility of a quantum operation are discussed in the context of q -entropies. By the monotonicity property, the lower bound of Pinsker type in terms of the trace norm distance is obtained for the Tsallis relative q -entropy of order $q = 1/2$. For $0 \leq q \leq 2$, Fano type quantum inequalities with freely variable parameters are obtained.

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I. INTRODUCTION

In both the classical and quantum information theory, the Fano inequality is one of the key tools. It is essential to prove the converse to Shannon's second theorem [1]. The quantum Fano inequality is needed for complete proof of the quantum data processing inequality [2]. Some generalizations of the Shannon entropy have found use in various topics. One of frequently used entropic measures was proposed by Rényi [3]. Fano type inequalities in terms of Rényi's entropy are important in the context of classification problems [4]. Another variant of one-parametric extension was introduced in classical information theory by Havrda and Charvát [5] and in statistical physics by Tsallis [6]. The Tsallis entropy was found to be very significant in numerous topics of physics and other sciences [7]. In particular, Tsallis relative-entropy minimization can be applied to statistical inference problems [8, 9]. For $q > 1$, a q -parametric extension of the classical Fano inequality was given in [10]. The entropic uncertainty principle has been expressed in terms of both the Rényi [11, 12] and Tsallis entropies [13, 14].

Blahut showed that the standard Fano inequality can be derived from the properties of the relative entropy [15]. A development of this idea leads to a family of Fano-like inequalities for random variables [16]. The author of the paper [17] proposed extensions of quantum Fano's inequality on the base of monotonicity of the quantum relative entropy. For $0 \leq q \leq 2$, the Tsallis relative entropy also enjoys the monotonicity under the action of quantum operations. The aim of the present work is to examine Fano type quantum inequalities in terms of Tsallis' q -entropies. Inequalities of such a kind will be obtained on the base of monotonicity as well as in another way. We also discuss a connection between the monotonicity and lower bounds on the relative q -entropy. A Pinsker type lower bound is deduced for $q = 1/2$. The paper is organized as follows. In Section II, the definitions and preliminary results are presented. A generalization of the quantum Fano inequality in terms of Tsallis' entropies is obtained in Section III. Lower bounds on the relative q -entropy are considered in Section IV. In Section V, a family of Fano type quantum inequalities is obtained on the base of monotonicity property. Section VI concludes the paper with a summary of results.

II. DEFINITIONS AND NOTATION

First, we recall the definitions of used entropic measures. For real $q \geq 0$ and $q \neq 1$, we define the non-extensive q -entropy of probability vector $\mathbf{p} = (p_1, \dots, p_n)$ by [6]

$$S_q(\mathbf{p}) \triangleq (1 - q)^{-1} \left(\sum_{i=1}^n p_i^q - 1 \right) = \sum_{i=1}^n \eta_q(p_i), \quad (2.1)$$

where $\eta_q(x) = (x^q - x)/(1 - q)$ for brevity. This can be recast as $S_q(\mathbf{p}) = -\sum_i p_i^q \ln_q p_i$ in terms of the q -logarithm $\ln_q x \equiv (x^{1-q} - 1)/(1 - q)$, defined for $q \geq 0$, $q \neq 1$ and $x > 0$. The quantity (2.1) will be referred to as "Tsallis q -entropy", though it was previously discussed by Havrda and Charvát [5]. In the limit $q \rightarrow 1$, $\ln_q x \rightarrow \ln x$ and

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the quantity (2.1) recovers the Shannon entropy. For any $p \in [0; 1]$, the binary Tsallis entropy is defined as $H_q(p) = \eta_q(p) + \eta_q(1-p)$. The entropy (2.1) reaches the maximal value $\ln_q n$ with the uniform distribution $p_i = 1/n$. For normalized density operator ρ on d -dimensional Hilbert space, the Tsallis q -entropy is defined as

$$S_q(\rho) \triangleq (1-q)^{-1} (\text{tr}(\rho^q) - 1) = \text{tr}(\eta_q(\rho)) . \quad (2.2)$$

The maximal value $\ln_q d$ is reached for maximally mixed state $\mathbb{1}/d$. The limit $q \rightarrow 1$ leads to the von Neumann entropy $S_1(\rho) = -\text{tr}(\rho \ln \rho)$. Its general properties are summarized in [18].

In the classical regime, the relative q -entropy was defined as [19]

$$D_q(\mathbf{p}||\mathbf{r}) \triangleq -\sum_i p_i \ln_q(r_i/p_i) = (1-q)^{-1} \left(1 - \sum_i p_i^q r_i^{1-q}\right) . \quad (2.3)$$

For basic properties of this measure, see Refs. [19, 20]. In particular, the relative entropy $D_q(\mathbf{p}||\mathbf{r})$ is monotone for all $q \geq 0$ [20]. Namely, if $\mathbf{T} = [[t_{ij}]]$ denotes the transition probability matrix, obeying $\sum_i t_{ij} = 1$ for all j , then

$$D_q(\mathbf{T}\mathbf{p}||\mathbf{T}\mathbf{r}) \leq D_q(\mathbf{p}||\mathbf{r}) \quad (0 \leq q) , \quad (2.4)$$

where probability vectors are put as columns. This fact easily follows from the generalized log-sum inequality derived in [19]. In the binary case, we will write

$$D_q(u, v) \equiv D_q(\{u, 1-u\}||\{v, 1-v\}) \quad (u, v \in [0; 1]) . \quad (2.5)$$

For $0 \leq q < 1$, a quantum extension seems to be obvious. If ρ and σ are normalized density operators then [20, 21]

$$D_q(\rho||\sigma) \triangleq (1-q)^{-1} (1 - \text{tr}(\rho^q \sigma^{1-q})) . \quad (2.6)$$

When $q > 1$, the case of singular σ should be taken into account. The expression (2.6) can be adopted for $\ker(\sigma) \subset \ker(\rho)$, otherwise $D_q(\rho||\sigma) = +\infty$. For $0 \leq q \leq 2$, the quantum relative q -entropy enjoys the monotonicity under trace-preserving quantum operations. The formalism of quantum operations provides a unified treatment of possible state change in quantum theory [2]. Let \mathcal{H} and \mathcal{H}' be finite-dimensional Hilbert spaces, and let operators E_μ map \mathcal{H} to \mathcal{H}' . Any trace-preserving quantum operation \mathcal{E} is represented as linear map [2]

$$\rho \mapsto \mathcal{E}(\rho) = \sum_m E_m \rho E_m^\dagger , \quad (2.7)$$

given that $\text{tr}(\mathcal{E}(\rho)) = 1$ for all normalized inputs ρ . The last condition is equivalent to $\sum_m E_m^\dagger E_m = \mathbb{1}$, where $\mathbb{1}$ is the identity operator on \mathcal{H} . The map (2.7) must be completely positive as well [2]. The monotonicity of quantum relative q -entropy implies that for any trace-preserving \mathcal{E} ,

$$D_q(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq D_q(\rho||\sigma) \quad (0 \leq q \leq 2) . \quad (2.8)$$

This inequality can be obtained by applying of Lieb's concavity theorem for $0 \leq q < 1$ and Ando's convexity theorem for $1 < q \leq 2$ (for a review of this issue, see [22]). It also follows from the general results of the papers [23, 24], since the function $x \mapsto x^q$ is matrix concave for $0 \leq q \leq 1$ and matrix convex for $1 \leq q \leq 2$ (see, e.g., chapter V of Ref. [25]).

The quantum Fano inequality imposes an upper bound on the entropy exchange in terms of the entanglement fidelity [26]. Here we deal with the two quantum systems, reference system R and principal system Q . The initial state ρ^Q of system Q is mapped into $\mathcal{E}^Q(\rho^Q)$. To monitor the entanglement transmission, we consider a purification $|\Psi^{RQ}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_Q$ which is transformed into the final state of the joint system RQ given by

$$\rho^{R'Q'} = \mathcal{I}^R \otimes \mathcal{E}^Q(|\Psi^{RQ}\rangle\langle\Psi^{RQ}|) . \quad (2.9)$$

The system R itself is not altered, i.e. $\text{tr}_Q(\rho^{R'Q'}) = \text{tr}_Q(|\Psi^{RQ}\rangle\langle\Psi^{RQ}|)$. The entanglement fidelity is defined as

$$F(\rho^Q, \mathcal{E}^Q) = \langle\Psi^{RQ}|\rho^{R'Q'}|\Psi^{RQ}\rangle . \quad (2.10)$$

In Ref. [26], Schumacher defined the entropy exchange as $S_1(\rho^Q, \mathcal{E}^Q) = S_1(R', Q') = -\text{tr}(\rho^{R'Q'} \ln \rho^{R'Q'})$. Imaging an environment E , we can reexpress the quantum operation \mathcal{E}^Q as

$$\mathcal{E}^Q(\rho^Q) = \text{tr}_E \left(\mathbb{U}_{QE}(\rho^Q \otimes |e_0\rangle\langle e_0|) \mathbb{U}_{QE}^\dagger \right) . \quad (2.11)$$

Since the final state $(\mathbb{1}_R \otimes \mathbb{U}_{QE})|\Psi^{RQ}\rangle \otimes |e_0\rangle$ of the triple system RQE is obviously pure, the final density operators $\rho^{R'Q'}$ of RQ and $\rho^{E'}$ of E are both the partial traces of the same one-rank projector. So these final density operators have the same non-zero eigenvalues, whence the entropy exchange is equal to $S_1(E') = -\text{tr}(\rho^{E'} \ln \rho^{E'})$ [2]. Due to a similar observation, both the entanglement fidelity and entropy exchange are not dependent on the choice of initial purification $|\Psi^{RQ}\rangle$ [26]. We are now ready to quantify the entanglement transmission by other entropic functionals.

Definition II.1. For $q \geq 0$ and $q \neq 1$, the q -entropy exchange is defined by

$$S_q(\rho^Q, \mathcal{E}^Q) \triangleq S_q(R', Q') = \text{tr}(\eta_q(\rho^{R'Q'})) . \quad (2.12)$$

As non-zero eigenvalues of the operators $\rho^{R'Q'}$ and $\rho^{E'}$ are the same, we have $S_q(\rho^Q, \mathcal{E}^Q) = S_q(E') = \text{tr}(\eta_q(\rho^{E'}))$. So this quantity characterizes an amount of q -entropy introduced by the operation \mathcal{E}^Q into an initially pure environment E . Because the state of E after the action of \mathcal{E}^Q is [2]

$$\rho^{E'} = \sum_{mn} w_{mn} |e_m\rangle \langle e_n| , \quad (2.13)$$

where $w_{mn} = \text{tr}(\mathbf{E}_m \rho^Q \mathbf{E}_n^\dagger)$ are entries of the matrix \mathbf{W} , we have $S_q(\rho^Q, \mathcal{E}^Q) = \text{tr}(\eta_q(\mathbf{W}))$. In a similar manner, the entanglement fidelity can be expressed as [26]

$$F(\rho^Q, \mathcal{E}^Q) = \sum_m |\text{tr}(\rho^Q \mathbf{E}_m)|^2 . \quad (2.14)$$

The last formulae for the entropy exchange and the entanglement fidelity are rather useful for explicit calculations. Since the main definitions are already given, we may simplify the notation to $S_q(\rho, \mathcal{E})$ and $F(\rho, \mathcal{E})$. That is, we will omit the label " Q " whenever density matrices and quantum operations are related to the principal system Q solely.

The quantum q -entropy is concave for all $q \geq 0$ [27]. Because the matrix \mathbf{W} is linear in ρ , the q -entropy exchange $S_q(\rho, \mathcal{E})$ is therefore concave in the first argument ρ . Moreover, it is concave in the second argument, i.e.

$$\theta S_q(\rho, \mathcal{G}) + (1 - \theta) S_q(\rho, \mathcal{F}) \leq S_q(\rho, \theta \mathcal{G} + (1 - \theta) \mathcal{F}) \quad (2.15)$$

for any trace-preserving \mathcal{G}, \mathcal{F} and all $\theta \in [0; 1]$. Indeed, we have $\theta \rho_{\mathcal{G}}^{R'Q'} + (1 - \theta) \rho_{\mathcal{F}}^{R'Q'} = \rho_{\mathcal{E}}^{R'Q'}$ for quantum operation $\mathcal{E} = \theta \mathcal{G} + (1 - \theta) \mathcal{F}$. This claim follows from the linearity of each quantum operation and the representation (2.9). So the concavity of the q -entropy (2.2) provides (2.15).

III. QUANTUM FANO INEQUALITY FOR THE q -ENTROPY EXCHANGE

In this section, we obtain an upper bound on the q -entropy exchange in terms of the entanglement fidelity. The method of derivation is very direct in character and similar to the well-known proof of the standard Fano inequality. Before obtaining the main result, we briefly recall one auxiliary statement. Let $x \mapsto f(x)$ be a concave function of real scalar. Then for arbitrary Hermitian operator \mathbf{X} and arbitrary normalized state $|\psi\rangle$, there holds

$$\langle \psi | f(\mathbf{X}) | \psi \rangle \leq f(\langle \psi | \mathbf{X} | \psi \rangle) . \quad (3.1)$$

To prove the claim, we take the spectral decomposition $\mathbf{X} = \sum_j x_j |x_j\rangle \langle x_j|$, whence $f(\mathbf{X}) = \sum_j f(x_j) |x_j\rangle \langle x_j|$ and

$$\langle \psi | f(\mathbf{X}) | \psi \rangle = \sum_j |c_j|^2 f(x_j) . \quad (3.2)$$

Here the numbers $c_j = \langle x_j | \psi \rangle$ are related to the expansion $|\psi\rangle = \sum_j c_j |x_j\rangle$ and satisfy $\sum_j |c_j|^2 = 1$. By Jensen's inequality for the concave function f , we have

$$\sum_j |c_j|^2 f(x_j) \leq f\left(\sum_j |c_j|^2 x_j\right) = f(\langle \psi | \mathbf{X} | \psi \rangle) . \quad (3.3)$$

Incidentally, we can observe that the functional $\Phi(\mathbf{X}) = \text{tr}(f(\mathbf{X}))$ is concave as well, i.e.

$$\theta \Phi(\mathbf{Y}) + (1 - \theta) \Phi(\mathbf{Z}) \leq \Phi(\theta \mathbf{Y} + (1 - \theta) \mathbf{Z}) \quad (3.4)$$

for Hermitian \mathbf{Y}, \mathbf{Z} and all $\theta \in [0; 1]$. If the $|x_j\rangle$'s are eigenstates of $\mathbf{X} = \theta \mathbf{Y} + (1 - \theta) \mathbf{Z}$, then we actually get

$$\begin{aligned} \Phi(\mathbf{X}) &= \sum_j f(\langle x_j | \mathbf{X} | x_j \rangle) = \sum_j f\left(\theta \langle x_j | \mathbf{Y} | x_j \rangle + (1 - \theta) \langle x_j | \mathbf{Z} | x_j \rangle\right) \\ &\geq \sum_j \theta f(\langle x_j | \mathbf{Y} | x_j \rangle) + \sum_j (1 - \theta) f(\langle x_j | \mathbf{Z} | x_j \rangle) \end{aligned} \quad (3.5)$$

$$\geq \theta \sum_j \langle x_j | f(\mathbf{Y}) | x_j \rangle + (1 - \theta) \sum_j \langle x_j | f(\mathbf{Z}) | x_j \rangle , \quad (3.6)$$

or else $\Phi(\mathbf{X}) \geq \theta \text{tr}(f(\mathbf{Y})) + (1 - \theta) \text{tr}(f(\mathbf{Z}))$. Here the step (3.5) follows from the concavity of the function $f(x)$, the step (3.6) follows from (3.1). Since the function $\eta_q(x)$ is concave for $q \geq 0$, the above reasons show the concavity of the q -entropy. The desired upper bound on the q -entropy exchange is posed as follows.

Theorem III.1. For $q \geq 0$, the q -entropy exchange is bounded from above as

$$S_q(\rho, \mathcal{E}) \leq H_q(F(\rho, \mathcal{E})) + (1 - F(\rho, \mathcal{E}))^q \ln_q(d^2 - 1) . \quad (3.7)$$

Proof. Let $\{|i\rangle\}$ be an orthonormal basis for the system RQ such that $|1\rangle = |\Psi^{RQ}\rangle$. We will use (3.1), since the function $\eta_q(x)$ is concave for all $q \geq 0$. Introducing the operator

$$\mathbf{X}' = \sum_{i=1}^{d^2} |i\rangle\langle i| \rho^{R'Q'} |i\rangle\langle i| , \quad (3.8)$$

the numbers $r_i = \langle i| \rho^{R'Q'} |i\rangle$ are eigenvalues of \mathbf{X}' . Due to this fact and (3.1), we then obtain

$$S_q(\rho, \mathcal{E}) = \text{tr}(\eta_q(\rho^{R'Q'})) = \sum_{i=1}^{d^2} \langle i| \eta_q(\rho^{R'Q'}) |i\rangle \leq \sum_{i=1}^{d^2} \eta_q(r_i) = \text{tr}(\eta_q(\mathbf{X}')) , \quad (3.9)$$

where the right-hand side is the Tsallis q -entropy $S_q(\mathbf{r})$ of d^2 -dimensional probability vector $\mathbf{r} = (r_1, r_2, \dots, r_{d^2})$. We also note that $r_1 = F(\rho, \mathcal{E})$ by the choice of $|1\rangle$. Putting $b_i = (1 - r_1)^{-1} r_i$ for $2 \leq i \leq d^2$, we get

$$\begin{aligned} S_q(\mathbf{r}) &= \eta_q(r_1) - \sum_{i=2}^{d^2} (1 - r_1)^q b_i^q \ln_q((1 - r_1)b_i) \\ &= \eta_q(r_1) - (1 - r_1)^q \sum_{i=2}^{d^2} b_i^q \left(b_i^{1-q} \ln_q(1 - r_1) + \ln_q b_i \right) \\ &= \eta_q(r_1) + \eta_q(1 - r_1) - (1 - r_1)^q \sum_{i=2}^{d^2} b_i^q \ln_q b_i = H_q(r_1) + (1 - r_1)^q S_q(\mathbf{b}) , \end{aligned} \quad (3.10)$$

where we used the identity $\ln_q(xy) = y^{1-q} \ln_q x + \ln_q y$ and $\sum_{2 \leq i \leq d^2} b_i = 1$. The right-hand side of (3.10) does not exceed $H_q(r_1) + (1 - r_1)^q \ln_q(d^2 - 1)$, since the \mathbf{b} is a $(d^2 - 1)$ -dimensional probability vector. ■

Note that d^2 is replaced by $d_R d$, when the reference system R has a Hilbert space of dimension $d_R < d$ [26]. The relation (3.7) shows that if the q -entropy exchange is large then the entanglement fidelity should be small enough. The notion of mutual information is basic in classical information theory [1] and also used in some scenarios of quantum information [28–30]. In other aspects, a similar role is played by the quantum coherent information [2]

$$I_1(\rho, \mathcal{E}) = S_1(\mathcal{E}(\rho)) - S_1(\rho, \mathcal{E}) . \quad (3.11)$$

By analogy, we can define the coherent q -information as

$$I_q(\rho, \mathcal{E}) \triangleq S_q(\mathcal{E}(\rho)) - S_q(\rho, \mathcal{E}) . \quad (3.12)$$

The right-hand side of (3.12) looks similar to the f -generalization of the coherent information treated in [24]. But the above expression is actually not a partial case of such generalization. We now recall that the Tsallis q -entropy enjoys the subadditivity property for $q > 1$, namely

$$S_q(Q, E) \leq S_q(Q) + S_q(E) . \quad (3.13)$$

The inequality has been conjectured by Raggio [27] and later proved by Audenaert [31]. Raggio also conjectured that the inequality (3.13) is saturated if and only if either of the systems Q and E is being in a pure state, and proved this in a partial case. It seems that equality conditions for (3.13) are beyond the scope of the subadditivity proof given in [31]. Here the question to be answered is whether the equality in (3.13) implies that either of two subsystems is being in a pure state. Using (3.13), we can derive a triangle type inequality

$$|S_q(Q) - S_q(E)| \leq S_q(Q, E) \quad (1 < q) . \quad (3.14)$$

The proof is easy. Introducing the reference system R , one purifies systems Q and E . Due to (3.13), we then have

$$S_q(R, Q) \leq S_q(R) + S_q(Q) . \quad (3.15)$$

When state of the triple system RQE is pure, $S_q(R, Q) = S_q(E)$ and $S_q(R) = S_q(Q, E)$. These two equalities allows to rewrite (3.15) in form

$$S_q(E) - S_q(Q) \leq S_q(Q, E) . \quad (3.16)$$

By a parallel argument, we get $S_q(Q) - S_q(E) \leq S_q(Q, E)$, and the last two inequalities provide (3.14). This treatment allows further extension to many of the quantum unified entropies [32]. These entropies were introduced and motivated in [33]. We can now establish an upper bound on the coherent q -information.

Theorem III.2. *For $q > 1$, the coherent q -information is bounded from above by*

$$I_q(\rho, \mathcal{E}) \leq S_q(\rho) . \quad (3.17)$$

Assuming Raggio's conjecture on equality conditions in (3.13), the equality in (3.17) implies that the quantum operation \mathcal{E} is perfectly reversible upon input of ρ .

Proof. Using the definition (3.12) and the triangle inequality, we obtain

$$I_q(\rho, \mathcal{E}) = S_q(Q') - S_q(E') \leq S_q(Q', E') . \quad (3.18)$$

Note that the operation \mathcal{E} is realized by some unitary transformation of the space $\mathcal{H}_Q \otimes \mathcal{H}_E$ and the initial state of environment E is pure (see the formula (2.11)). These points imply the equality $S_q(Q', E') = S_q(Q, E) = S_q(\rho)$, which together with (3.18) provides (3.17). Assume that Raggio's conjecture on equality conditions holds. The equality in (3.17) can be rewritten as

$$S_q(R', E') = S_q(R') + S_q(E') \quad (3.19)$$

due to $S_q(Q') = S_q(R', E')$ and $S_q(Q', E') = S_q(R')$ (the final state of the triple system RQE is pure). So either of the systems R and E should be in a pure state, whence $\rho^{R'E'} = \rho^{R'} \otimes \rho^{E'}$ (for the last claim, see [27]). This product structure immediately implies an existence of the recovery operation \mathcal{R} such that the entanglement fidelity of combined operation $F(\rho, \mathcal{R} \circ \mathcal{E}) = 1$ (for an explicit construction of \mathcal{R} , see the proof of theorem 12.10 in [2]). In other words, the quantum operation \mathcal{E} is perfectly reversible upon input of ρ . ■

So the coherent q -information enjoys, in a less degree, similar properties to the coherent information (3.11). The standard data processing inequality also tells that for $q = 1$ the perfect reversibility of \mathcal{E} upon input of ρ leads to the equality in (3.17). This statement is based on the quantum Fano inequality and the strong subadditivity property [2]. In the classical regime, the Tsallis entropy of order $q > 1$ obeys the strong subadditivity [10]. However, this result cannot be used, since the systems R and E become entangled after action of the operation \mathcal{E} . Using another definition, the author of [24] has extended the data processing inequality in complete setting to a wide class of matrix convex functions. Note that the function $x \mapsto x^q$ does enjoy the matrix convexity for $1 \leq q \leq 2$, but does not for $2 < q$ (see, e.g., exercise V.2.11 in [25]). On the other hand, Theorem III.2 holds for all $1 < q$. In general, possible ways to extend the standard concept of coherent information deserve further investigations.

IV. NOTES ON PINSKER TYPE INEQUALITIES

Lower and upper bounds on some functional allows to estimate it in terms of other measures or parameters. When states are close to each other in the trace norm sense, corresponding bounds characterize continuity of a functional. Estimates of such a kind are important due to a statistical interpretation of the trace distance in terms of POVM measurements [2]. The partitioned trace distances also enjoy this property for one-rank POVMs [34]. The well-known upper bound of desired type is given by Fannes' inequality for the von Neumann entropy [35]. This treatment has been extended to the Tsallis q -entropy [36, 37] and its partial sums [38]. For the standard relative entropy, lower and upper continuity bounds are obtained in the paper [39]. For the relative q -entropy, some upper continuity bounds were given in [40]. The well-known Pinsker type lower bound is expressed as

$$D_1(\rho||\sigma) \geq \frac{1}{2} \|\rho - \sigma\|_1^2 , \quad (4.1)$$

where the Schatten 1-norm is $\|X\|_1 = \text{tr} \sqrt{X^\dagger X}$ for any operator X . In much more general setting, this inequality was proved in [41]. It is also known [42] that

$$D_p(\rho||\sigma) \geq D_1(\rho||\sigma) \geq D_q(\rho||\sigma) , \quad (4.2)$$

where $q \in [0; 1)$ and $p \in (1; 2]$. So the upper bounds given in [39] hold for the relative q -entropy of order $q \in [0; 1)$, the lower ones hold for the relative q -entropy of order $q \in (1; 2]$. Thus, we are rather interested in lower bounds for the former and in upper bounds for the latter. Upper continuity bounds on the relative q -entropy of order $q \in (1; 2]$ have recently been obtained in [40]. Below we will discuss lower continuity bounds that follow from the monotonicity of the relative q -entropy.

Theorem IV.1. Let Π_+ be a projector on the eigenspace corresponding to positive eigenvalues of the difference $(\rho - \sigma)$. For $q \in [0; 2]$ and any pairs of density operators, the relative q -entropy is bounded from below as

$$D_q(\rho||\sigma) \geq D_q(u, v) , \quad (4.3)$$

where $u = \text{tr}(\Pi_+ \rho)$ and $v = \text{tr}(\Pi_+ \sigma)$.

Proof. Let us write the Jordan decomposition of traceless Hermitian operator

$$\rho - \sigma = \sum_{r>0} r |r\rangle\langle r| - \sum_{s>0} s |s\rangle\langle s| . \quad (4.4)$$

We define the two projectors $\Pi_+ = \sum_r |r\rangle\langle r|$, $\Pi_- = \sum_s |s\rangle\langle s|$. When the difference $(\rho - \sigma)$ has zero eigenvalues, corresponding eigenvectors should be included to the orthonormal sets $\{|r\rangle\}$ and $\{|s\rangle\}$ anyhow; then $\Pi_+ + \Pi_- = \mathbb{1}$. Consider the trace-preserving quantum operation

$$\mathcal{F}(\rho) = \sum_r |r\rangle\langle r| \rho |r\rangle\langle r| + \sum_s |s\rangle\langle s| \rho |s\rangle\langle s| = \sum_r u_r |r\rangle\langle r| + \sum_s u_s |s\rangle\langle s| , \quad (4.5)$$

where probabilities $u_r = \langle r|\rho|r\rangle$ and $u_s = \langle s|\rho|s\rangle$. Putting $v_r = \langle r|\sigma|r\rangle$ and $v_s = \langle s|\sigma|s\rangle$, we also write

$$\mathcal{F}(\sigma) = \sum_r v_r |r\rangle\langle r| + \sum_s v_s |s\rangle\langle s| . \quad (4.6)$$

So the outputs $\mathcal{F}(\rho)$ and $\mathcal{F}(\sigma)$ are diagonal in the same basis. Due to this fact and the monotonicity of quantum relative q -entropy for $0 \leq q \leq 2$, we have

$$D_q(\rho||\sigma) \geq D_q(\mathcal{F}(\rho)||\mathcal{F}(\sigma)) = D_q(\{u_r, u_s\}||\{v_r, v_s\}) . \quad (4.7)$$

We shall again use the monotonicity, but now in classical regime. Let us put the 2-by- d transition probability matrix

$$\mathbb{T} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix} , \quad (4.8)$$

in which the units of the first row act on r -components, the units of the second row act on s -components. This matrix maps the distributions $\{u_r, u_s\}$ and $\{v_r, v_s\}$ to $\{u, 1-u\}$ and $\{v, 1-v\}$ respectively with $u = \sum_r u_r = \text{tr}(\Pi_+ \rho)$, $v = \sum_r v_r = \text{tr}(\Pi_+ \sigma)$. By the monotonicity, the right-hand side of (4.7) is not less than $D_q(u, v)$. ■

In general, the projector Π_+ and the probabilities u, v are not uniquely defined. But for any choice, we have $u - v = \sum_r r = (1/2) \|\rho - \sigma\|_1$. The next stage is to estimate $D_q(u, v)$ from below in terms of the quantity $|u - v|$. Really, we would like to find the minimum of $D_q(u, v)$ under the conditions $0 \leq u \leq 1$, $0 \leq v \leq 1$ and $|u - v| = t$. For the standard case $q = 1$, this issue is well developed (see [43] and references therein). A complete examination of the problem would take us to far afield. We consider only the case $q = 1/2$, which allows simple calculations.

Lemma IV.2. In the domain $\{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1, |u - v| = t \in (0; 1)\}$, there holds

$$g(u, v) = \sqrt{uv} + \sqrt{(1-u)(1-v)} \leq \sqrt{1-t^2} . \quad (4.9)$$

Proof. The domain consists of the two segments, the segment $u = t + v$ with $v \in [0; 1-t]$ and the segment $v = t + u$ with $u \in [0; 1-t]$. Due to symmetry, we consider the former. By differentiating with respect to v under the constraint $u = t + v$, we obtain the condition for critical points in a form

$$\frac{v+u}{2\sqrt{uv}} = \frac{(1-v) + (1-u)}{2\sqrt{(1-u)(1-v)}} , \quad (4.10)$$

which is clearly satisfied with $u + v = 1$. Combining this with $u = t + v$ gives $u_0 = (1+t)/2$, $v_0 = (1-t)/2$, and $g(u_0, v_0) = \sqrt{1-t^2}$. Some inspection shows that the above critical point is unique on the chosen segment (the solution $u = v$ of (4.10) holds only for $t = 0$). The value $g(u_0, v_0)$ is actually maximal, since the function is concave and $g(t, 0) = g(1, 1-t) = \sqrt{1-t}$. ■

Using the relations $\mathcal{D}_{1/2}(u, v) = 2(1 - g(u, v))$, $t = (1/2)\|\rho - \sigma\|_1$, and $1 - \sqrt{1-t^2} \geq t^2/2$, we finally get

$$\mathcal{D}_{1/2}(\rho||\sigma) \geq 2 - 2 \left(1 - \frac{1}{4} \|\rho - \sigma\|_1^2 \right)^{1/2} \geq \frac{1}{4} \|\rho - \sigma\|_1^2 . \quad (4.11)$$

This is a quantum lower bound of Pinsker type on the relative 1/2-entropy. As expressed in terms of the trace norm distance, it characterizes a continuity property. The lower bounds (4.1) and (4.11) are independent and consistent in view of (4.2). In a similar manner, lower bounds of Pinsker type could be obtained for other values from the interval $q \in (0; 1)$. By the statement of Theorem IV.1, the problem is merely reduced to minimization of $D_q(u, v)$ under certain conditions. Except for the case $q = 1/2$, an answer is not so obvious. In principle, this issue might be a subject of separate research.

V. FANO TYPE INEQUALITIES FOR $0 \leq q \leq 2$

In this section, a family of Fano type bounds on the q -entropy exchange will be derived from the monotonicity of relative q -entropy. In the regular case $q = 1$, this idea has been developed for the classical Fano inequality [16] as well as for the quantum one [17]. The basic point is to relate the q -entropy exchange with the relative q -entropy by

$$D_q(\rho^{R'Q'} || \tilde{\Omega}) = -S_q(R', Q') - \text{tr} \left((\rho^{R'Q'})^q \ln_q(\tilde{\Omega}) \right) , \quad (5.1)$$

which follows from the identity $(1-q)^{-1} (x - x^q y^{1-q}) = -\eta_q(x) - x^q \ln_q y$ and the normalization. By $\tilde{\Omega}$ we denote an arbitrary nonsingular density matrix of appropriate dimensionality. The q -entropy exchange is bounded from above in the following way.

Theorem V.1. *Let $\tilde{\Omega}$ be a nonsingular density matrix on the space $\mathcal{H}_R \otimes \mathcal{H}_Q$, $|\Psi^{RQ}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_Q$ a purification of the input ρ of the operation \mathcal{E} , $F_e = F(\rho, \mathcal{E})$, and $F_\Omega = \langle \Psi^{RQ} | \tilde{\Omega} | \Psi^{RQ} \rangle$. For $0 \leq q \leq 2$, there holds*

$$S_q(\rho, \mathcal{E}) \leq -\mathcal{D}_q(F_e, F_\Omega) - \text{tr} \left((\rho^{R'Q'})^q \ln_q(\tilde{\Omega}) \right) . \quad (5.2)$$

Proof. Let $\{|i\rangle\}$ be an orthonormal basis in $\mathcal{H}_R \otimes \mathcal{H}_Q$ such that $|1\rangle = |\Psi^{RQ}\rangle$. We consider the trace-preserving quantum operation \mathcal{G} acting as

$$\mathcal{G}(\rho^{R'Q'}) = \sum_{i=1}^{d^2} |i\rangle \langle i| \rho^{R'Q'} |i\rangle \langle i| , \quad (5.3)$$

$$\mathcal{G}(\tilde{\Omega}) = \sum_{i=1}^{d^2} |i\rangle \langle i| \tilde{\Omega} |i\rangle \langle i| . \quad (5.4)$$

Both the above outputs are diagonal in the basis $\{|i\rangle\}$. Combining this fact with the monotonicity for $q \in [0; 2]$, we further write

$$D_q(\rho^{R'Q'} || \tilde{\Omega}) \geq D_q \left(\mathcal{G}(\rho^{R'Q'}) || \mathcal{G}(\tilde{\Omega}) \right) = D_q(\mathbf{r} || \mathbf{w}) , \quad (5.5)$$

where the probabilities $r_i = \langle i | \rho^{R'Q'} | i \rangle$ and $w_i = \langle i | \tilde{\Omega} | i \rangle$. We now apply the monotonicity in classical regime with the 2-by- d^2 transition probability matrix such that

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix} , \quad \mathbf{T}\mathbf{r} = \begin{pmatrix} r_1 \\ 1 - r_1 \end{pmatrix} , \quad \mathbf{T}\mathbf{w} = \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix} . \quad (5.6)$$

By the monotonicity, we then get $D_q(\mathbf{r} || \mathbf{w}) \geq \mathcal{D}_q(F_e, F_\Omega)$ in view of $r_1 = F_e$ and $w_1 = F_\Omega$. Combining this with the relations (5.1) and (5.5) finally gives (5.2). ■

In the right-hand side of (5.2), the first term is negative, the second is positive. The relation (5.2) with freely variable terms presents some family of upper bounds on the q -entropy exchange. Choosing various forms of $\tilde{\Omega}$, we will be arrived at different upper bounds. A nonlinear power q of the density operator $\rho^{R'Q'}$ is difficult to any transformations when this operator is not given in an explicit form. Nevertheless, for $q \geq 1$ there holds

$$(\rho^{R'Q'})^q \leq \rho^{R'Q'} , \quad (5.7)$$

since eigenvalues of a density matrix does not exceed one. We also have $-\ln_q(\tilde{\Omega}) \geq \mathbf{0}$, due to $\ln_q x \leq 0$ for $x \in (0; 1]$. Because $\mathbf{X} \leq \mathbf{Y}$ implies $\text{tr}(\mathbf{X}\mathbf{A}) \leq \text{tr}(\mathbf{Y}\mathbf{A})$ for any $\mathbf{A} \geq \mathbf{0}$, the inequalities (5.2) and (5.7) lead to

$$S_q(\rho, \mathcal{E}) \leq -\mathcal{D}_q(F_e, F_\Omega) - \text{tr} \left(\rho^{R'Q'} \ln_q(\tilde{\Omega}) \right) \quad (1 \leq q \leq 2) . \quad (5.8)$$

If the principal system Q is initially prepared in the state

$$\rho = \sum_{k=1}^d \lambda_k |\lambda_k\rangle \langle \lambda_k| , \quad (5.9)$$

then any purification has the form

$$|\Psi^{RQ}\rangle = \sum_{k=1}^d \sqrt{\lambda_k} |\xi_k\rangle \otimes |\lambda_k\rangle \quad (5.10)$$

with some orthonormal basis $\{|\xi_k\rangle\}$ in \mathcal{H}_R . For a probability distribution $\{\mu_j\}$, we can take

$$\tilde{\Omega} = \sum_{j=1}^d \mu_j |\xi_j\rangle\langle\xi_j| \otimes \omega = \sum_{jk} \mu_j \nu_k |\xi_j \nu_k\rangle\langle\xi_j \nu_k|, \quad (5.11)$$

where $\omega = \sum_k \nu_k |\nu_k\rangle\langle\nu_k|$ is a density operator on \mathcal{H}_Q . By calculations, one obtains

$$F_{\Omega} = \sum_{ijk} \sqrt{\lambda_i \lambda_k} \mu_j \langle\xi_i|\xi_j\rangle \langle\xi_j|\xi_k\rangle \langle\lambda_i|\omega|\lambda_k\rangle = \sum_{j=1}^d \lambda_j \mu_j \langle\lambda_j|\omega|\lambda_j\rangle. \quad (5.12)$$

Using the identity $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$, we also find

$$\ln_q(\tilde{\Omega}) = \sum_{jk} \ln_q(\mu_j \nu_k) |\xi_j \nu_k\rangle\langle\xi_j \nu_k| = \sum_j \ln_q \mu_j |\xi_j\rangle\langle\xi_j| \otimes \mathbb{1}_Q + \sum_j \mu_j^{1-q} |\xi_j\rangle\langle\xi_j| \otimes \ln_q(\omega). \quad (5.13)$$

Using (2.9) and the linearity of \mathcal{E} , we further obtain

$$\rho^{R'Q'} = \sum_{ij} \sqrt{\lambda_i \lambda_j} |\xi_i\rangle\langle\xi_j| \otimes \mathcal{E}(|\lambda_i\rangle\langle\lambda_j|). \quad (5.14)$$

We also observe that $\text{tr}(\mathcal{E}(|\phi\rangle\langle\psi|)) = \langle\psi|\phi\rangle$ by the preservation of the trace. Hence the trace of the product of (5.13) and (5.14) is written as

$$\text{tr}(\rho^{R'Q'} \ln_q(\tilde{\Omega})) = \sum_j \lambda_j \ln_q \mu_j + \sum_j \mu_j^{1-q} \lambda_j \text{tr}(\mathcal{E}(|\lambda_j\rangle\langle\lambda_j|) \ln_q(\omega)), \quad (5.15)$$

where both the $\{\mu_j\}$ and ω are still arbitrary. Combining (5.12) and (5.15) with (5.8), we obtain an upper bound of Fano type, in which both the probability distribution $\{\mu_j\}$ and state ω are freely variable.

A next question is, whether the q -parametric extension of Fano inequality (3.7) can be derived from (5.2). It seems that the answer is negative in general. For $q = 1$, the quantum Fano inequality is get by $\tilde{\Omega} = (\mathbb{1}_R \otimes \mathbb{1}_Q)/d^2$ (for details, see [17]). For $q > 1$, such a choice leads to an inequality which includes the right-hand side of (3.7) with some additional terms. However, these terms are not negative anywhere. We refrain from presenting the calculations here. Moreover, any corollary of (5.2) would be restricted to $q \in [0, 2]$, whereas the inequality (3.7) holds for all $q \geq 0$. On the other hand, the relation (5.2) with freely variable parameters may lead to new inequalities similar to (5.8). The results of this section are essentially based on the monotonicity of the relative q -entropy for $0 \leq q \leq 2$. We finally note that the classical Fano inequality deals with the conditional entropy [1], which is not a direct classical analog of the quantum entropy exchange. So the quantum formulation enough differs from the classical one. In this regard, any extension of results of the papers [15, 16] to generalized entropic functionals would be interesting.

VI. CONCLUSIONS

We have considered various extensions of the quantum Fano inequality in terms of q -entropic measures. The notion of the q -entropy exchange was introduced with some discussion of its properties. In particular, the q -entropy exchange is concave in the input density matrix as well as in the running quantum operation. The standard quantum Fano inequality is generalized for all $q \geq 0$ in Theorem III.1. This result is essentially based on the properties of the function $\eta_q(x)$ and the related functional inequality (3.1). We have also introduced a q -parametric extension of the coherent information. Due to the subadditivity for $q > 1$, the triangle inequality (3.14) holds. Using this result, the upper bound on the coherent q -information is posed in Theorem III.2. Assuming Raggio's conjecture, the inequality (3.17) is saturated only if the quantum operation \mathcal{E} is perfectly reversible upon input of ρ .

We have also obtained some bounds based on the monotonicity of the relative q -entropy for $0 \leq q \leq 2$. For all $q \in [0, 2]$, a simple lower bound on the relative q -entropy is established by Theorem IV.1. Hence a lower continuity bound of Pinsker type has been obtained for $q = 1/2$ in the result (4.11). An extension to other values of parameter q is briefly discussed. The monotonicity property has been used for obtaining a family of Fano type quantum inequalities on the q -entropy exchange. This statement is formulated in Theorem V.1. Except for $q = 1$, the inequality of Theorem III.1 seems to be not included in the presented family. Nevertheless, several interesting inequalities with freely variable parameters can be dealt. These inequalities might be useful in specialized problems, when some prior knowledge on both the input state and running quantum operation is available.

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